Chapter 2: Analytic Functions

Definition A function $f : \Omega \to \mathbb{C}$ is a rule that assigns to each $z \in \Omega \subseteq \mathbb{C}$ a unique complex number $f(z) \in \mathbb{C}$. The set Ω is the domain of definition of f. If $S \subseteq \Omega$, then the set

$$f(S) \overset{\text{def}}{=} \{ f(z) : z \in S \}$$

is the image of S. The set $f(\Omega)$ is the range of f. Points in $f(\Omega)$ are called values of f. //

If $f : \Omega \to \mathbb{C}$ is a function, then the value $f(x+iy) = u + iv$ depends on $(x,y) \in \mathbb{R}^2$. Collecting all values, we decompose f into real and imaginary parts:

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$ are real valued functions of 2 real variables. Sometimes we will write

$$\operatorname{Re} f = u \quad \text{and} \quad \operatorname{Im} f = v. \qquad //$$

Example Some examples of functions.

(1) $f(z) = z^2$. Let $z = x+iy$. then

$$f(z) = (x+iy)(x+iy) = x^2 - y^2 + i(2xy).$$

So, $u(x,y) = x^2 + y^2$ and $v(x,y) = 2xy$.

(2) $f(z) = |z|^2 = x^2 + y^2$ so $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$.
Such a function is real-valued.

(3) Polynomials are functions of the form

$$P(z) = a_0 + a_1 z + \dots + a_n z^n$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$. Then the degree of P is n.

(4) Rational functions are functions of the form $P(z)/Q(z)$

where $p(z), Q(z)$ are polynomials. The domain of definition is wherever $Q(z) \neq 0$.

(5) If we use polar coordinates for z, then a function f can be written

$$f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta).$$

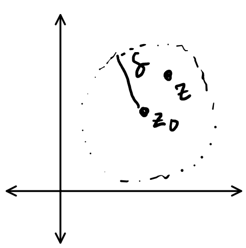
E.g. $f(z) = z^2$. If $z = re^{i\theta}$, then

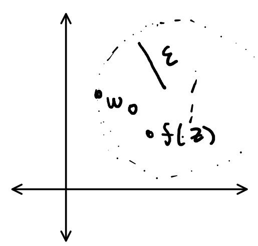
$$f(z) = r^2 e^{i2\theta} = \underbrace{r^2 \cos 2\theta}_{u(r,\theta)} + i\underbrace{r^2 \sin 2\theta}_{v(r,\theta)}.$$

(6) $f(z) = z^{1/n}$. This is not a function at all! We saw last time that $z^{1/n}$ has n distinct values. Such a "function" is called multiple-valued. We can make this into a single-valued function by assigning a single value of $z^{1/n}$ for each z. For instance, taking the principal nth root of z.

Limits of Functions

Definition (Limit of a function) Suppose f is defined in a deleted neighborhood of $z_0 \in \mathbb{C}$. We say that the limit of f as z approaches z_0 is $w_0 \in \mathbb{C}$ if: for all $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $|f(z) - w_0| < \varepsilon$.





In this case we write

$$\lim_{z \to z_0} f(z) = w_0.$$

Intuitively, the limit of f at z_0 is w_0 if we can make $f(z)$ arbitrarily close to w_0 by taking z sufficiently close to z_0.

Example We show that $\lim z^2 = -1$ using the definition.

$\boxed{\text{Example}}$ We show that $\lim_{z \to i} z^2 = -1$ using the definition.

Proof. Let $\varepsilon > 0$. Note $|z^2 - (-1)| = |z-i||z+i|$. First suppose $0 < |z-i| < 1$. Then

$$\begin{aligned} |z+i| &= |z-i+2i| \\ &\le |z-i| + |2i| \\ &= \underbrace{|z-i|}_{<1} + 2 < 3. \end{aligned}$$

Choose $\delta = \min\{\frac{\varepsilon}{3}, 1\}$. Then $0 < |z-i| < \delta$ implies

$$|z^2 - (-1)| = |z-i||z+i| < \frac{\varepsilon}{3} \cdot 3 = \varepsilon.$$

$\blacksquare$

$\boxed{\text{Theorem (Limits are Unique)}}$ If f has a limit at z_0, then it is unique.

Proof. Assume $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} f(z) = W_0$.

Let $\varepsilon > 0$. Choose $\delta_1 > 0$ such that

$$0 < |z - z_0| < \delta_1 \implies |f(z) - w_0| < \varepsilon/2.$$

Choose $\delta_2 > 0$ such that

$$0 < |z - z_0| < \delta_2 \implies |f(z) - W_0| < \varepsilon/2.$$

Now, if $0 < |z - z_0| < \min\{\delta_1, \delta_2\}$, then

$$\begin{aligned} |w_0 - W_0| &= |w_0 - f(z) + f(z) - W_0| \\ &\le |w_0 - f(z)| + |f(z) - W_0| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary

$$|w_0 - W_0| = 0.$$

Hence, $w_0 = W_0$.

$\blacksquare$

Limits can also be taken if z_0 is a boundary point of a region R on which f is defined. We simply require that the inequality

$$0 < |z - z_0| < \delta$$

be satisfied only for points belonging to R <u>and</u> a deleted neighborhood of z_0.

Uniqueness of limits can be used to show that a limit does not exist.

$\boxed{\text{Example}}$ The function $f(z) = \dfrac{z}{\bar{z}}$ has no limit at 0.

Let $z = x + i$, then $f(z) = \dfrac{x+iy}{x-iy}$.

Along the real axis, $\text{Im}\, z = 0$ so $z = x$. So $f(z) = \dfrac{x}{x} = 1$.
Along the imaginary axis, $\text{Re}\, z = 0$ so $z = iy$. So $f(z) = \dfrac{iy}{-iy} = -1$.
Taking the limit along these axes, gives different values. By uniqueness of limits, the limit doesn't exist. //

Theorems on Limits

$\boxed{\text{Theorem (Limits in Terms of Re f / Im f)}}$ Suppose that

$$f(z) = f(x+iy) = u(x,y) + i v(x,y).$$

Then

$$\lim_{x+iy \to x_0+iy_0} f(x+iy) = u_0 + i v_0$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. Choose $\delta > 0$ such that

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$0 < |x+iy - (x_0 + iy_0)| < \delta \implies |u(x,y) + iv(x,y) - (u_0 + iv_0)| < \varepsilon.$$

Notice that $$|x+iy - (x_0+iy_0)| = \underbrace{\sqrt{(x-x_0)^2 + (y-y_0)^2}}_{\text{distance between } (x,y),\,(x_0,y_0)}$$

Notice also that

$$|u(x,y) - u_0| \leq |u(x,y) + iv(x,y) - (u_0+iv_0)| < \varepsilon \quad \text{and}$$

$$|v(x,y) - v_0| \leq |u(x,y) + iv(x,y) - (u_0+iv_0)| < \varepsilon.$$

This proves the claim.

$\blacksquare$

$\boxed{\text{Theorem (Limit Laws)}}$ Suppose

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z\to z_0} F(z) = W_0.$$

Then (1) $\displaystyle \lim_{z\to z_0} \big(f(z) + F(z)\big) = w_0 + W_0$

(2) $\displaystyle \lim_{z\to z_0} f(z)F(z) = w_0 W_0$

(3) $\displaystyle \lim_{z\to z_0} f(z)/F(z) = \frac{w_0}{W_0}$, if $W_0 \neq 0$.

Proof. Use preceding theorem together with limit laws from caclulus.

$\blacksquare$

$\boxed{\text{Example}}$ If $p(z)$ is a polynomial, then

$$\lim_{z\to z_0} p(z) = p(z_0).$$

Write $p(z) = a_0 + a_1 z + \cdots + a_n z^n$. By the limit laws:

$$\lim_{z\to z_0} p(z) \overset{(1)}{=} \lim_{z\to z_0} a_0 + \lim_{z\to z_0} a_1 z + \cdots + \lim_{z\to z_0} a_n z^n$$

$$\overset{(2)}{=} \lim_{z\to z_0} a_0 + \lim_{z\to z_0} a_1 \lim_{z\to z_0} z + \cdots + \lim_{z\to z_0} a_n \lim_{z\to z_0} z^n$$

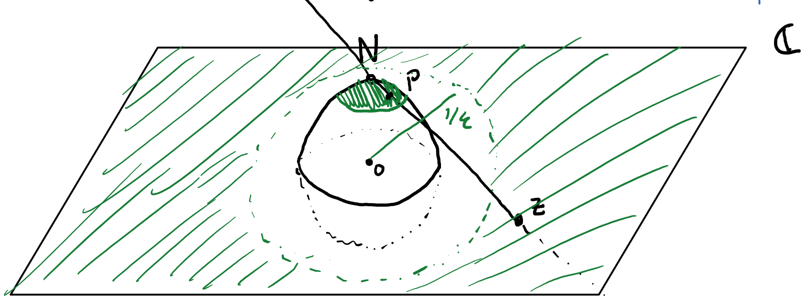
$$= a_0 + a_1 \lim z + \cdots + a_n \lim z^n$$

$$= a_0 + a_1 \lim_{z\to z_0} z + \cdots + a_n \lim_{z \to z_0} z^n$$

Also $\lim_{z\to z_0} z^n = z_0^n$ by induction. $\to\ = a_0 + a_1 z_0 + \cdots + a_n z_0^n$
$= p(z_0)$. //

Definition (Extended Complex Plane / Riemann Sphere)

The extended complex plane is the set $\mathbb{C}$ together with a symbol ∞ called the point at infinity. There is a one-to-one correspondence between the extended complex plane and the unit sphere given by stereographic projection.



The point N corresponds to ∞ and any point P on the sphere corresponds to a unique point $z \in \mathbb{C}$ lying at the intersection of $\mathbb{C}$ and the line between P and N.

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Definition (Neighborhood of infinity) Let $\varepsilon > 0$. The set

$$\{ z \in \mathbb{C} : |z| > 1/\varepsilon \}$$

is called a neighborhood of ∞. Geometrically, a neighborhood of infinity is the exterior of a circle $C_{1/\varepsilon}(0)$, which corresponds to a neighborhood of N on the Riemann sphere. //

Meaning is easily given to limits

$$\lim_{z \to z_0} f(z) = w_0$$

(Limits involving ∞)

where z_0 or w_0 are allowed to be ∞. We simply replace the appropriate neighborhood in the original definition with neighborhoods of ∞.

Theorem (Limits Involving Infinity)

(1) If $\lim_{z \to z_0} 1/f(z) = 0$, then $\lim_{z \to z_0} f(z) = \infty$.

(2) If $\lim_{z \to 0} f(1/z) = w_0$, then $\lim_{z \to \infty} f(z) = w_0$.

(3) If $\lim_{z \to 0} 1/f(1/z) = 0$, then $\lim_{z \to \infty} f(z) = \infty$.

Proof.

(1) Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \text{implies} \quad \left| \frac{1}{f(z)} - 0 \right| < \varepsilon.$$

The condition $\left|\frac{1}{f(z)}\right| < \varepsilon$ is the same as $\frac{1}{\varepsilon} < |f(z)|$, i.e. $f(z)$ is in a neighborhood of ∞.

(2) Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$0 < |z - 0| < \delta \quad \text{implies} \quad |f(1/z) - w_0| < \varepsilon.$$

The condition $0 < |z| < \delta$ is equivalent to $\frac{1}{\delta} < \left|\frac{1}{z}\right|$. Replacing $1/z$ w/ z we get

$$1/\delta < |z| \quad \text{implies} \quad |f(z) - w_0| < \varepsilon.$$

■

Example $\lim_{z \to \infty} \dfrac{2z^4+1}{z^3+1} = \infty$. Using (3), we compute

$$\lim_{z \to 0} \frac{(1/z)^3+1}{2(1/z)^4+1} = \lim_{z \to 0} \frac{1+z^3}{z^3} \Big/ 2\left(\frac{1+z^4}{z^4}\right) = \lim_{z \to 0} \frac{(1+z^3)\cdot z^4}{2(1+z^4)\, z^3} = \lim_{z \to 0} \frac{(1+z^3)\, z}{2(1+z^4)} = 0. \; //$$

Continuous Functions

Definition (Continuous functions) A function f is continuous at z_0 if
$$\lim_{z \to z_0} f(z) = f(z_0).$$
That is: for all $\varepsilon > 0$, $\exists \delta > 0$ such that
$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon.$$
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By the limit laws, sums, products, and quotients of continuous functions are continuous (where they are defined).

Theorem (Composition of continuous functions) Suppose that f is defined on an open disk $D_\varepsilon(z_0)$ and the domain of g contains $f(D_\varepsilon(z_0))$. If f is continuous at z_0 and g is continuous at $f(z_0)$, then $g \circ f$ is continuous at z_0.

Proof. Let $\varepsilon > 0$. Choose $\delta_1 > 0$ such that
$$|w - f(z_0)| < \delta_1 \implies |g(w) - g(f(z_0))| < \varepsilon.$$
Choose $\delta_2 > 0$ such that
$$|z - z_0| < \delta_2 \implies |f(z) - f(z_0)| < \delta_1$$
Now, $|z - z_0| < \delta_2$ implies $|g(f(z)) - g(f(z_0))| < \varepsilon$.

■

Theorem If f is continuous and nonzero at z_0, then there exists $\varepsilon > 0$ such that $f(z) \neq 0$ for all $z \in D_\varepsilon(z_0)$.

Proof. Suppose f is continuous and nonzero at z_0. Then $|f(z_0)| > 0$. Take $\varepsilon = \frac{|f(z_0)|}{2}$. Suppose $f(z) = 0$ for some $z \in D_\varepsilon(z_0)$. Then by continuity at z_0,
$$0 < |f(z_0)| = |f(z) - f(z_0)| < \varepsilon = \frac{|f(z_0)|}{2}.$$
Contradiction!

■

Theorem (Continuity in Terms of Re f/Im f) Suppose that
$$f(z) = u(x,y) + i\,v(x,y).$$

Then f is continuous at $z_0 = x_0 + iy_0$ if and only if both u and v are continuous at (x_0, y_0).

Proof. Follows from theorem on limits in terms of $\operatorname{Re} f / \operatorname{Im} f$. ∎

A subset of $\mathbb{C}$ is compact if it is closed and bounded. A function $f : \Omega \xrightarrow{\subseteq \mathbb{C}} \mathbb{C}$ is bounded if there exists $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in \Omega$.

Theorem (Extreme Value theorem) If R is a compact set and $f : R \to \mathbb{C}$ is continuous on R, then f is bounded and it achieves this bound.

Proof. If $f = u(x,y) + i\,v(x,y)$ is continuous, then $u, v : R \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}$ are continuous on R. Hence, so is

$$|f(z)| = \sqrt{u(x,y)^2 + v(x,y)^2}.$$

By vector calc, $|f(z)|$ is bound and achives its bound. ∎

Differentiable Functions

Definition (Derivative) Suppose the domain of definition of f contains an open disk $D_\varepsilon(z_0)$. The derivative of f at z_0 is the limit

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

When the limit exists, f is differentiable. Letting $\Delta z = z - z_0$, this can also be written

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

Example $f(z) = z^2$. We have

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0} 2z + \Delta z$$
$$= 2z.$$

//

Sometimes it will be convenient to use the notation
$$\Delta w = f(z+\Delta z) - f(z)$$
So that
$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

$\boxed{\text{Example}}$ Where is $f(z) = |z|^2$ differentiable? Let $z \in \mathbb{C}$.

Compute
$$\Delta w = |z+\Delta z|^2 - |z|^2$$
$$= (z+\Delta z)\overline{(z+\Delta z)} - |z|^2$$
$$= \cancel{z\bar{z}} + z\overline{\Delta z} + \Delta z\,\bar{z} + \Delta z\,\overline{\Delta z} - \cancel{|z|^2}$$
$$= z\overline{\Delta z} + \Delta z\,\bar{z} + \Delta z\,\overline{\Delta z}.$$

Then
$$\frac{\Delta w}{\Delta z} = \frac{z\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}.$$

Along the real axis $\Delta z = \overline{\Delta z}$. So
$$\frac{\Delta w}{\Delta z} = z + \bar{z} + \overline{\Delta z}$$
as $\Delta z \to 0$ the limit is $z + \bar{z}$. Along the imaginary axis, $\Delta z = -\overline{\Delta z}$ so
$$\frac{\Delta w}{\Delta z} = \bar{z} - z - \Delta z.$$
As $\Delta z \to 0$ the limit is $\bar{z} - z$. Since limits are unique, if $f'(z)$ exists, then $\bar{z} - z = z + \bar{z}$. Hence $z = 0$. Does $f'(0)$ exist? When $z = 0$
$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0.$$

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The preceding example shows two surprising facts:

(1) f' can exist at a single point and nowhere else

in a neighborhood of that point.

(2) Re f / Im f can have continuous partial derivatives of all orders, and yet f' does not exist.

Note: $\mathrm{Re}(|f|^2) = x^2+y^2$ and $\mathrm{Im}(|f|^2) = 0$.

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$\boxed{\text{Proposition (Differentiable functions are Continuous)}}$ If f is differentiable at z_0, then f is continuous at z_0.

Proof. Suppose f is differentiable at z_0. Then

$$\lim_{z\to z_0}\left(f(z)-f(z_0)\right) = \left(\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}\right)\underbrace{\left(\lim_{z\to z_0} z-z_0\right)}_{=0} = 0.$$

Hence, $\lim_{z\to z_0} f(z) = f(z_0)$.

$\blacksquare$

$\boxed{\text{Proposition (Differentiation Laws)}}$ Suppose f and g are differentiable at z. Then

(1) $\frac{d}{dz} c = 0$, $\forall c \in \mathbb{C}$

(2) $\frac{d}{dz}(c f(z)) = c f'(z)$, $\forall c \in \mathbb{C}$ (Constant Rule)

(3) $\frac{d}{dz} z^n = n z^{n-1}$, $\forall n \in \mathbb{N}$ (Power Rule)

(4) $\frac{d}{dz}(f(z)+g(z)) = f'(z) + g'(z)$ (Sum Rule)

(5) $\frac{d}{dz}(f(z)g(z)) = f'(z)g(z) + g'(z)f(z)$ (Product Rule)

(6) $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{f'(z)g(z) - g'(z)f(z)}{g(z)^2}$, $g(z) \neq 0$ (Quotient Rule)

Proof.

(5) Compute

$$\begin{aligned}\Delta w &= f(z+\Delta z)g(z+\Delta z) - f(z)g(z)\\ &= f(z+\Delta z)g(z+\Delta z) - f(z+\Delta z)g(z) + f(z+\Delta z)g(z) - f(z)g(z)\\ &= f(z+\Delta z)\left(g(z+\Delta z)-g(z)\right) + g(z)\left(f(z+\Delta z)-f(z)\right).\end{aligned}$$

So $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \underbrace{\lim_{\Delta z \to 0} f(z+\Delta z)}_{= f(z) \text{ since } f \text{ is continuous}} \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z}$

$+ \lim_{\Delta z \to 0} g(z) \lim_{\Delta z \to 0} \frac{f(z+\Delta z - f(z)}{\Delta z}$

$= f(z)\, g'(z) + g(z)\, f'(z).$

$\blacksquare$

$\boxed{\text{Proposition (Chain Rule)}}$ Suppose that f is differentiable at z_0 and g is differentiable at $f(z_0)$. Then $g \circ f$ is differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0).$$

Proof. Since $g'(f(z_0))$ exists, there is open disk $D_\varepsilon(f(z_0))$ on which g is defined. On this disk define a function

$$\Phi(w) = \frac{g(w) - g(f(z_0))}{w - f(z_0)} - g'(f(z_0)). \quad (*)$$

Notice $\lim_{w \to f(z_0)} \Phi(w) = 0$. Hence, $\Phi(w)$ is continuous at $f(z_0)$.

Rewrite $(*)$ as

$$(*) \quad g(w) - g(f(z_0)) = \Big(\Phi(w) + g'(f(z_0))\Big)\big(w - f(z_0)\big).$$

Now, since f is continuous at z_0 choose $\delta > 0$ such that

$|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \varepsilon$, that is $f(z) \in D_\varepsilon(f(z_0))$.

Now, when $|z - z_0| < \delta$, we can take $w = f(z)$ in $(*)$ and divide by $z - z_0$:

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \big(\Phi(f(z)) + g'(f(z_0)\big) \left(\frac{f(z) - f(z_0)}{z - z_0}\right).$$

Taking the limit $z \to z_0$ we get

$$g'(f(z)) = \big(0 + g'(f(z_0))\big)\, f'(z_0)$$
$$= g'(f(z_0))\, f'(z_0).$$

$\blacksquare$

Cauchy-Riemann Equations

Now, we derive the Cauchy-Riemann Partial differential equations. Suppose that $f(z) = u(x,y) + i\,v(x,y)$ and that $f'(z)$ exists. Writing $z = x+iy$ and $\Delta z = \Delta x + i\Delta y$, we compute:

$$\frac{\Delta w}{\Delta z} = \frac{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - (u(x,y) + i\,v(x,y))}{\Delta x + i\Delta y}$$

$$= \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\left(\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}\right).$$

Along the real axis, $\Delta y = 0$ so we get

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= u_x(x,y) + i\,v_x(x,y).$$

Along the imaginary axis, $\Delta x = 0$ so we get

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$= \frac{1}{i} u_y + \frac{i}{i} v_y = v_y(x,y) - i\,u_y(x,y) \quad \left(\frac{1}{i} = -i\right)$$

So, since limits are unique we get

$$u_x + i v_x = v_y - i u_y$$

Hence, compare Re/Im part:

$$\boxed{\begin{cases} u_x(x,y) = v_y(x,y) \\ u_y(x,y) = -v_x(x,y) \end{cases} \quad \text{Cauchy-Riemann Equations}}$$

□

We just proved:

Theorem (Cauchy-Riemann Equations) Suppose that

$$f(z) = u(x,y) + i\,v(x,y)$$

is differentiable at $z = x+iy$. Then

(1) the first order partial derivatives of u and v exist and satisfy the Cauchy-Riemann Equations

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

(2) $f'(z) = u_x(x,y) + i\, v_x(x,y) = v_y(x,y) - i\, v_x(x,y)$. //

The CR-equations are a necessary condition for f' to exist. We can use them to locate some points where the derivative does not exist.

$\boxed{\text{Example}}$ $f(z) = |z|^2 = x^2+y^2 + i\,0$. Note $u(x,y) = x^2+y^2$ and $v(x,y)=0$,

We have $u_x(x,y) = 2x \qquad v_x(x,0) = 0$

$u_y(x,y) = 2y \qquad v_y(0,0) = 0$

The CR-Riemann equations:

$$2x = u_x = v_y = 0$$

$$2y = u_y = -v_x = 0,$$

So $x=0$ and $y=0$.

So $f'(z)$ does not exist when $z \neq 0$. //

Note: this doesn't show that $f'(0)$ exists.

The Cauchy-Riemann Equations are not sufficient for the existence of the derivative, as the next example shows.

$\boxed{\text{Example}}$ Suppose $f(z) = \begin{cases} \dfrac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$. Then

$$u(x,y) = \begin{cases} \dfrac{x^3-3xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \quad \text{and} \quad v(x,y) = \begin{cases} \dfrac{y^3-3x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

We show that u,v satisfy the CR-eq at 0.

$$u_x(0,0) = \lim_{\Delta x \to 0} \frac{u(0+\Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta x^3}{\Delta x^2} - 0}{\Delta x} = 1$$

$$u_y(0,0) = \lim_{\Delta y \to 0} \frac{u(0, 0+\Delta y) - u(0,0)}{\Delta y} = 0.$$

$$v_x(0,0) = \lim_{\Delta x \to 0} \frac{v(0+\Delta x, 0) - v(0,0)}{\Delta x} = 0$$

$$v_y(0,0) = \lim_{\Delta y \to 0} \frac{v(0,0+\Delta y) - v(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{\Delta y^3}{\Delta y^2} - 0}{\Delta y} = 1.$$

Hence
$$u_x(0,0) = 1 = v_y(0,0)$$
$$u_y(0,0) = 0 = -0 = -v_x(0,0)$$

So CR-eq are satisfied. But $f'(0)$ does not exist (exercise). //

$\boxed{\text{Theorem (Sufficient Condition for Differentiability)}}$ Suppose that $f(z) = u(x,y) + i v(x,y)$ is defined on a neighborhood of $z = x+iy$.

If (1) the first order partial derivatives of u, v exist everywhere in the neighborhood;

(2) the partial derivatives are continuous and satisfy the CR-equations at (x,y);

then $f'(z)$ exists and is given by $f'(z) = u_x(x,y) + i v_x(x,y)$.

Proof. See the book. $\square$

Example

$x \in \mathbb{R}$, so e^x is the usual exponential

$iy \notin \mathbb{R}$ so e^{iy} is defined by Euler's formular

(1) $f(z) = e^x e^{iy} = e^x \cos y + i e^x \sin y$. Note that $u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$ have continuous partial derivatives on all of $\mathbb{R}^2$. Moreover
$$u_x = e^x \cos y = v_y$$
$$u_y = -e^x \sin y = -v_x.$$

So CR-eq are satisfied everywhere $\Rightarrow f'(z)$ exists everywhere on $\mathbb{C}$.

(2) $f(z) = |z|^2 = x^2 + y^2 + i \cdot 0$. Note $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. These have continuous first order partial derivatives in a neighborhood of 0:
$$u_x = 2x \qquad v_x = 0$$
$$u_y = 2y \qquad v_y = 0$$

The Cauchy Riem. eq's are satisfied at 0, so $f'(0) = 0$ exists. //

The necessary and sufficient conditions can be translated into polar coordinates. If $f(z) = u(r,\theta) + i\, v(r,\theta)$, then the polar form of the CR-equations at $z = re^{i\theta}$ is

$$\boxed{\begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases} \quad \text{(evaluated at } (r,\theta))}$$

and the value of $f'(z)$ is

$$f'(re^{i\theta}) = e^{-i\theta}\left(u_r(r,\theta) + i\, v_r(r,\theta)\right).$$

The precise statements of these theorems are in the book. //

Example Consider

$$f(z) = \sqrt{r}\, e^{i\frac{\theta}{2}}, \quad (r > 0, -\pi < \theta < \pi)$$

This is the function that takes values as the principal square root of z. We compute f' using the sufficient condition in polar form. Notice

$$f(z) = \underbrace{\sqrt{r}\cos \theta/2}_{u(r,\theta)} + i\underbrace{\sqrt{r}\sin \theta/2}_{v(r,\theta)}.$$

Note

$$r\, u_r(r,\theta) = r\frac{1}{2\sqrt{r}}\cos \theta/2 = v_\theta(r,\theta)$$

$$u_\theta(r,\theta) = -\frac{1}{2}\sqrt{r}\sin \theta/2 = -r\, v_r(r,\theta)$$

So Cauchy-Riemann are satisfied everywhere and the partial are continuous everywhere. Then //

$$\begin{aligned} f' &= e^{-i\theta}\left(u_r + i v_r\right) \\ &= e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos \theta/2 + i\frac{1}{2\sqrt{r}}\sin \theta/2\right) \\ &= \frac{1}{2\sqrt{r}}e^{-i\theta}\left(\cos \theta/2 + i \sin \theta/2\right) = \frac{1}{2\sqrt{r}}e^{-i\theta/2} = \frac{1}{2f(z)}. \end{aligned}$$

Analytic Functions

Definition (Analytic) A function f is analytic on an open set U if $f'(z)$ exists for all $z \in U$. We say that f is analytic at a point z_0 if it is analytic some open disk $D_\varepsilon(z_0)$. A function is entire if it is analytic on $\mathbb{C}$. //

Example

(1) $f(z) = 1/z$ is analytic on any open set not containing 0, in particular on $\mathbb{C}\setminus\{0\}$.

(2) $f(z) = |z|^2$ is not analytic anywhere since we showed that $f'(z)$ exists if and only if $z=0$.

(3) Polynomials are entire. //

Let D be a domain. We know several necessary or sufficient for a function f to be analytic on D:

(Necessary) (1) f is continuous on all of D.

(2) Cauchy - Riemann eq. satisfied on D.

(sufficient) (1) Cauchy-Riemann eq satisfied on D and the partial derivatives of u and v are continuous on all of D.

(2) The differentiation rules. If f and g are analytic on D, then so are $f+g$, fg, f/g ($g\neq 0$ on D).

(3) The composition of analytic functions is analytic.

Theorem (sufficient condition for f to be constant) Suppose that D is a domain and $f'(z) = 0$ for all $z \in D$. Then $f(z)$ is constant on D.

Proof. Write $f(z) = u(x,y) + i\,v(x,y)$. We have

$$0 = f'(z) = u_x + i v_x$$
$$0 = f'(z) = v_y - i u_y$$

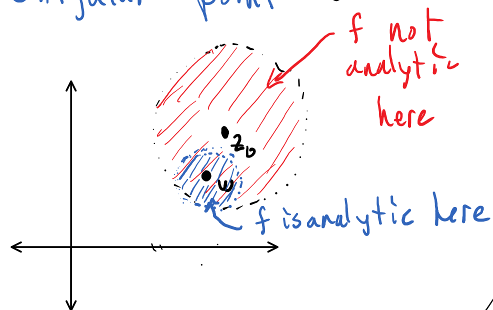
Hence, $u_x = u_y = 0$ and $v_x = v_y = 0$. Let L be any line segment connected points $p, q \in D$. Let $\vec{w} = (a, b)$ be a unit vector parallel to L. The directional derivative of u along L is

$$\nabla u \cdot \vec{w} = a u_x + b u_y = 0.$$

So u is constant on L. Since D is a domain, any two points can be connected by a polygonal line. Apply this argument to each line segment to see that u has the the same value at the endpoints. This shows that u is constant on D. The same argument works for v. Hence, if $u(x,y) = c$ and $v(x,y) = d$, then

$$f(z) = c + id.$$

$\boxed{\text{Definition (Singularities)}}$ If f is not analytic at z_0, but every neighborhood of z_0 contains a point at which f is analytic, then z_0 is called a singular point or singularity of f.



$\boxed{\text{Example}}$

(1) $f(z) = 1/z$ has a singularity at 0.

(2) $f(z) = |z|^2$ has no singular points. f is not analytic at

0, but any open neighborhood of 0 contains no points at which f is analytic.

(3) $f(z) = \dfrac{z^2+3}{(z+1)(z^2+5)}$ has singularities when

$$(z+1)(z^2+5) = 0$$

i.e. if $z = -1, i\sqrt{5}, -i\sqrt{5}$.

(4) $f(z) = \sin x \cosh y + i \cos x \sinh y$ is entire and hence has no singularities. In fact, f has derivatives of all orders.

Note that $u(x,y) = \sin x \cosh y$ and $v(x,y) = \cos x \sinh y$.

Recall:

$$\begin{cases} \sinh y = \dfrac{e^y - e^{-y}}{2} & \dfrac{d}{dy}\sinh y = \cosh y \\ \cosh y = \dfrac{e^y + e^{-y}}{2} & \dfrac{d}{dy}\cosh y = \sinh y. \end{cases}$$

Then

$$u_x(x,y) = \cos x \cosh y = v_y$$
$$u_y(x,y) = \sin x \sinh y = -v_x$$

So the CR-eq are satisfied everywhere and the partials are continuous everywhere. So f' exists and

$$f'(z) = u_x + i v_x = \underbrace{\cos x \cosh y}_{U(x,y)} - i \underbrace{\sin x \sinh y}_{V(x,y)}.$$

Then

$$U_x(x,y) = -\sin x \cosh y = V_y$$
$$U_y(x,y) = \cos x \sinh y = -V_x.$$

So $f''(z)$ exists and

$$f''(z) = U_x + i V_x = -\sin x \cosh y - i \cos x \sinh y$$
$$= -f(z).$$

//

$\boxed{\text{Proposition}}$ Suppose f and $\bar{f}$ are analytic on a domain D. Then f is constant on D.

Proof. Write

$$f(z) = u(x,y) + i v(x,y)$$
$$\bar{f}(z) = u(x,y) - i v(x,y).$$

Then $f, \bar{f}$ satisfy CR-eq on D:

for f : $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ for $\bar{f}$: $\begin{cases} u_x = -v_y \\ u_y = v_x \end{cases}$

Hence, $u_x = v_y = -u_x$ so $u_x = 0$. Also, $v_x = -u_y = -v_x$, so $v_x = 0$. Now, $f'(z) = u_x + i v_x = 0$. By the theorem, f is constant on D. ∎

$\boxed{\text{Proposition}}$ Suppose that f is analytic on a domain D and $|f(z)|$ is constant on D. Then f is constant on D.

Proof. Assume

$$(*) \quad |f(z)| = c \qquad \text{for all } z \in D.$$

Then

$$(*) \quad f(z)\bar{f}(z) = |f(z)|^2 = c^2 .$$

Suppose $c = 0$. Then by $(*)$ $f = 0$ so f is constant. If $c \neq 0$, then by $(*)$ $f(z) \neq 0$ for all $z \in D$. So then

$$\bar{f}(z) = \frac{c^2}{f(z)} .$$

But $\frac{c^2}{f(z)}$ is analytic since f is. Hence, $\bar{f}$ is analytic. Hence, f is constant by the preceding proposition. ∎